Towards finding the single-particle content of two-dimensional adjoint QCD

Uwe Trittmann

Department of Physics
Otterbein University
Westerville, OH 43081, USA

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Abstract

The single-particle content of two-dimensional adjoint QCD remains elusive due to the inability to distinguish single- from multi-particle states. To find a criterion we compare several approximations to the theory. Starting from the asymptotic theory (no pair production, only singular operators), we construct sets of eigenfunctions in the lowest parton sectors of the theory. A perturbative treatment of the omitted operators is performed. We find that multi-particle states are absent if pair-production is disallowed and hints for a double Regge trajectory of single-particle states. We discuss the structure of the eigensystem of the theory, and present the reason for the fact that bosonic single-particle states do not form multi-particle states.

1 Introduction

Two-dimensional Yang-Mills theory coupled to fermions in the adjoint representation, QCD_{2A}, has been discussed extensively in the literature [1, 3, 4, 9, 10], due to its many interesting features (see, e.g. [6]). However, its single-particle spectrum remains elusive, largely because there is no clear criterion to help purge the theory of its multi-particle content. Recently, QCD_{2A} has been numerically solved as a fermion theory. In [10], the authors use an idea from holography, namely that the theory is a trivial CFT in the UV limit. Therefore a decoupling between the low-lying spectrum and the high scaling-dimension quasi-primary operators ensues. A basis of these operators is constructed and cut off at a maximal (scaling) dimension. Good agreement is found with previous DLCQ results [4, 9, 12]. While [10] furnishes an important contribution to the ongoing debate over the single-particle content of QCD_{2A}, the disappointing conclusion is that also this approach is riddled with multi-particle states.

We present some progress on teasing out the true (single-particle) content of the theory described in more detail in Sec. 2. We start in Sec. 3 by considering the asymptotic approach of [1], in which the theory is solved for high excitation numbers, *i.e.* in a regime where parton number is conserved. We then explore the impact of non-singular interactions on the spectrum in Sec. 4. Sec. 5 is devoted to the role of the pair-production operators and the emergence of multi-particle states. Finally, we take a look at the implications of bosonization in Sec. 6 and conclude.

2 The Spectrum of QCD_{2A}

Adjoint QCD_2 is based on the following Lagrangian in light-cone coordinates $x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$, where x^+ plays the role of a time

$$\mathcal{L} = Tr[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma_\mu D^\mu \Psi], \tag{1}$$

where $\Psi = 2^{-1/4}\binom{\psi}{\chi}$, with ψ and χ being $N_c \times N_f$ matrices. The field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$, and the covariant derivative is defined as $D_\mu = \partial_\mu + i[A_\mu, \cdot]$. Working in the light-cone gauge, $A^+ = 0$, is consistent if the fermionic zero modes are omitted. The left-moving fermions can be integrated out, and the light-cone momentum P^+ and Hamiltonian P^- can be written in terms of the Fourier oscillation modes of the right-moving fermion only [4, 13]. Once the theory is formulated in terms of independent degrees of freedom, we can quantize it by imposing canonical anti-commutation relations at equal light-cone times x^+

$$\big\{\psi_{ij}(x^-), \psi_{kl}(y^-)\big\} = \frac{1}{2}\,\delta(x^- - y^-)\big(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\big). \tag{2}$$

One uses the usual decomposition of the fields in terms of fermion operators

$$\psi_{ij}(x^-) = \frac{1}{2\sqrt{\pi}}\int_0^\infty dk^+ \left(b_{ij}(k^+)\mathrm{e}^{-ik^+x^-} + b^\dagger_{ji}(k^+)\mathrm{e}^{ik^+x^-}\right) \ , \tag{3}$$

with anti-commutation relations following from Eq. (2)

$$\{b_{ij}(k^+), b^\dagger_{lk}(p^+)\} = \delta(k^+ - p^+)(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}) \tag{4}$$

to write the operators in terms of oscillators

$$\begin{aligned} P^+ &= \int_0^\infty dk\ k\, b^\dagger_{ij}(k) b_{ij}(k)\ , &(5)\\ P^- &= \frac{m^2}{2}\int_0^\infty \frac{dk}{k} b^\dagger_{ij}(k) b_{ij}(k) + \frac{g^2 N}{\pi}\int_0^\infty \frac{dk}{k}\ C(k) b^\dagger_{ij}(k) b_{ij}(k) &(6)\\ &+\frac{g^2}{2\pi}\int_0^\infty dk_1 dk_2 dk_3 dk_4 \Big\{ B(k_i)\delta(k_1+k_2+k_3-k_4) \\ &\qquad \times(b^\dagger_{kj}(k_4) b_{kl}(k_1) b_{li}(k_2) b_{ij}(k_3) - b^\dagger_{kj}(k_1) b^\dagger_{jl}(k_2) b^\dagger_{li}(k_3) b_{ki}(k_4)) \\ &\quad +A(k_i)\delta(k_1+k_2-k_3-k_4) b^\dagger_{kj}(k_3) b^\dagger_{ji}(k_4) b_{kl}(k_1) b_{li}(k_2) \\ &\quad +\frac{1}{2}D(k_i)\delta(k_1+k_2-k_3-k_4) b^\dagger_{ij}(k_3) b^\dagger_{kl}(k_4) b_{il}(k_1) b_{kj}(k_2) \Big\} \end{aligned}$$

with

$$\begin{aligned} A(k_i) &= \frac{1}{(k_4-k_2)^2} - \frac{1}{(k_1+k_2)^2}\ , &(7)\\ B(k_i) &= \frac{1}{(k_2+k_3)^2} - \frac{1}{(k_1+k_2)^2}, &(8)\\ C(k) &= \int_0^k dp\ \frac{k}{(p-k)^2}, &(9)\\ D(k_i) &= \frac{1}{(k_1-k_4)^2} - \frac{1}{(k_2-k_4)^2}, &(10) \end{aligned}$$

where the trace-splitting term $D(k_i)$ can be omitted at large N_c, and the trace-joining term is proportional to $B(k_i)$. The structure of the Hamiltonian P^- displayed in Eq. (6) is

$$P^- = P^-_m + P^-_{ren} + P^-_{PC,s} + P^-_{PC,r} + P^-_{PV} + P^-_{finiteN}. \tag{11}$$

Obviously, the mass term P^-_m is dropped in the massless theory, yet the renormalization operator P^-_{ren} needs to be included. Interactions that violate parton number, P^-_{PV}, couple blocks of different parton number, whereas parton-number conserving interactions P^-_{PC} are block diagonal, and may include singular(s) or regular(r) functions of the parton momenta.

If one considers large excitation numbers, parton-number violating operators proportional to $B(k_i)$ can be neglected and the mass of the fermions becomes irrelevant [1]. We will refer to the resulting approximation as the *asymptotic theory*: we retain the most singular terms in the interaction only, and additionally use the approximation

$$\int_0^1 \frac{dy}{(x-y)^2}\phi(y) \approx \int_{-\infty}^\infty \frac{dy}{(x-y)^2}\phi(y), \tag{12}$$

because for the highly excited states the integral is dominated by the interval around $x = y$, associated with the long-range Coulomb-type force. Thus, the asymptotic theory is split into decoupled sectors with fixed parton numbers subject to 't Hooft-like equations

$$M^2\phi_r(x_1, \ldots, x_r) = -\sum_{i=1}^{r}(-1)^{(r+1)(i+1)}\int_{-\infty}^{\infty}\frac{\phi_r(y, x_i + x_{i+1} - y, x_{i+2}, \ldots, x_{i+r-1})}{(x_i - y)^2}dy, \tag{13}$$

where the wavefunctions ϕ_r distribute momentum in the states of definite parton number r

$$|\Phi_r\rangle = \left(\prod_{j=1}^{r}\int_0^1 dx_j\right)\delta(1 - \sum_{i=1}^{r}x_i)\phi_r(x_1, x_2, \ldots, x_r)\frac{1}{N_c^{r/2}}Tr[b(-x_1)\cdots b(-x_r)]|0\rangle. \tag{14}$$

The x_i are momentum fractions with $\sum_i x_i = 1$, and the total momentum has been set to unity. The number of partons r is even (odd) for bosonic (fermionic) states.

A complete set of solutions of Eq. (13) remains elusive, while Ref. [1] displays what looks like half of the bosonic eigenfunctions, *i.e.* even-r eigenfunctions with eigenvalues $(-1)^{r/2+1}$ under the theory's Z_2 orientation symmetry

$$\mathcal{T} : b_{ij} \to b_{ji}. \tag{15}$$

In these sectors, the eigenfunctions listed in [1] have eigenvalues

$$M^2_{n_1,\ldots,n_k} = 2g^2N\pi^2(n_1 + n_2 + \cdots + n_k), \tag{16}$$

where the excitation numbers n_i are even and their sum is much larger than $k \equiv r/2$. This implies an exponentially growing density of states, and points towards the existence of a Hagedorn transition of the theory at high temperatures. Eq. (16) suggests a $r/2$-dimensional manifold of solutions in the r-parton sector. However, the $r-1$ relative momenta of the sector lead one to expect $r-1$ quantum numbers. Incidentally, an $r/2$-dimensional manifold of solutions makes it hard to think of a generalization to the fermionic (odd r) sectors of the theory. The functions displayed in [1] are therefore likely particular solutions; the general solutions should exhibit additional excitation numbers.

The clear separation of the eigenvalues, Eq. (16), does not guarantee that these are single-particle solutions. We know from [9] that exact and approximate multi-particle states exist in the single-trace sector of the theory, so that single-particle states cannot be identified with single-trace states. The problem is compounded by the approximations made. While omitting the non-singular terms in the interaction and discarding parton-changing operators can be justified on physical grounds, approximating the integral as in Eq. (12) implies unphysical effects which paradoxically make the solutions simpler. Furthermore, the correct generalization of 't Hooft's approximations [2] to higher parton sectors is a restriction of the Hilbert space from the naive $[0, 1]^r$ hypercube to a $(r-1)$-simplex, which takes up $1/r!$ of the former's volume, see Appendix. We expect fewer linearly independent eigensolutions on the simplex than on the hypercube.

In fact, multi-particle states (identified by their threshold masses) are absent altogether in the asymptotic theory. A quick DLCQ calculation traces this behavior back to the absence of parton-number violation. This means that a method to distinguish single- from multi-particle states cannot emerge from the asymptotic theory alone. Identifying threshold mass values as in [9] is not going to be good enough either: the alleged multi-particle states fulfill a single-particle integral equation[1]. On the other hand, one knows from the bosonized theory that states absent in the adjoint and identity block of the current algebra are true multi-particle states [5], and one can study them. The opposite is not true, and one has to learn how to identify the single-particle states in these current blocks. Unfortunately, it is unlikely that approximate solutions á la 't Hooft [2] and Kutasov [1] exist in the bosonized theory, because bosonization implies parton-number violation.

The eigenvalue problem at hand is equivalent to an integral equation which is completely specified *ab ovo*. As such, Eq. (13) implies that its solutions fulfill several constraints: the (pseudo-)cyclicity of the wavefunction

$$\phi_r(x_1, x_2, \ldots, x_r) = (-1)^{r+1}\phi_r(x_2, x_3 \ldots, x_r, x_1), \tag{17}$$

since the fermions are real, and the constraint

$$\phi_n(0, x_2, \ldots, x_n) = 0, \tag{18}$$

necessary to secure hermiticity of the Hamiltonian (only) in the presence of a mass term[2]. In the case of the 't Hooft model [2], this amounts to a "boundary condition" in the sense that the values of the wavefunction are specified at the endpoints of the interval. We find it advantageous to realize (and in some sense relax) the latter constraint by replacing it with the condition

$$\phi_n(x_1, x_2, \ldots, x_n) = \pm\phi_n(1 - x_1, 1 - x_2, \ldots, 1 - x_n), \tag{19}$$

which allows for a natural interpretation of the massless (massive) theory's solutions as (anti-)periodic functions. Of course, all constraints are fixed by the form of the integral equation, and cannot to be confused with the conditions specified to solve a differential equation. For instance, hermiticity is given, the vanishing of the wavefunctions follows.

Solutions of definite $\mathcal{T}$-symmetry, Eq. (15), fulfill an additional condition, which means that the wavefunctions have different support. Namely, some combination of creation operators might not exist in one symmetry sector. For example, in the four-parton sector a constraint arises because states like $Tr[b(-x)b(-x)b(-y)b(-y)]|0\rangle$ are $\mathcal{T}$ even. Analogous requirements exist in other sectors[3], except for the fermionic $\mathcal{T} = (-1)^{(r+1)/2}$ sectors[4].

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[1] I am grateful to D.G. Robertson for pointing this out.

[2] The apparent vanishing of the $\mathcal{T} = (-1)^{r/2}[(-1)^{(r-1)/2}]$ wavefunctions for even [odd] r at the ends of the intervals, see Fig. 1, is not due to a boundary condition but accidental; the computer code happens to choose $|1, 1, \ldots, K-r-1\rangle$ as first, and $|K/r, K/r, \ldots, K/r\rangle$ (or similar) as last basis state. At these points the eigenfunctions vanish due to symmetry constraints.

[3] For the first few parton sectors they are: $\phi_{3-}(x, x, y) = 0$, $\phi_{4+}(x, y, x, y) = 0$, $\phi_{4-}(x, x, y, y) = 0$, $\phi_{5+}(x, y, y, x, z) = 0$, $\phi_{6+}(x, y, y, x, z, z) = 0$, $\phi_{6-}(x, y, z, w, z, y) = 0$, and cyclic.

[4] Incidentally, these sectors sport a massless state when the above approximations are used.

3 Solving the asymptotic eigenvalue problem

We can solve the Kutasov integral equation (4.10) of Ref. [1] algebraically by using the following ansatz for the wavefunctions

$$|n_1, n_2, \dots n_{r-1}\rangle \doteq \prod_j^{r-1} e^{i\pi n_j x_j} = \phi_r(x_1, x_2, \dots, x_r), \tag{20}$$

where r is the number of partons, $x_r = 1 - \sum_j^{r-1} x_j$. Note that we have $r-1$ excitation numbers n_i, as expected from $r-1$ relative momenta in the r parton sector. The $r=2$ version solves the 't Hooft equation

$$\frac{M^2}{g^2 N_c} e^{i\pi n x} = -\int_{-\infty}^{\infty} \frac{dy}{(x-y)^2} e^{i\pi n y} = \pi^2 |n| e^{i\pi n x}. \tag{21}$$

In other words, we use the single-particle states of a Hamiltonian appropriate for the problem to construct a Fock basis, in the spirit of Ref. [8]. These single-particle states are two-parton states, and they constitute an orthonormal basis on the interval $[0,1]$. However, the multi-parton states live in a restricted Hilbert space because the total momentum is fixed, see Appendix. Clearly, Eq. (21) is insensitive to the sign of n. Hence, we admit positive and negative excitation numbers: $n_i \in 2\mathbb{Z}$ or $2\mathbb{Z}+1$.

There is a rather elegant solution to the eigenvalue problem, Eq. (13), based on the observation that the solutions of the adjoint 't Hooft problem have to be (anti-)cyclic, *cf.* Eq. (17). By introducing the cyclic permutation operator

$$\mathcal{C} : (x_1, x_2, \dots, x_r) \to (x_2, x_3, \dots, x_r, x_1),$$

we can construct the solution to the asymptotic adjoint 't Hooft problem by symmetrizing our ansatz

$$|n_1, n_2, \dots n_{r-1}\rangle_{sym} \equiv \frac{1}{\sqrt{r}} \sum_{k=1}^{r} (-1)^{(r-1)(k-1)} \mathcal{C}^{k-1} |n_1, n_2, \dots n_{r-1}\rangle, \tag{22}$$

where $\mathcal{C}^0 = 1$. This furnishes a general asymptotic solution of adjoint QCD_2. It is not hard to show that the eigenvalues are

$$M^2 = g^2 N \pi^2 \sum_{k=1}^{r} |n_1^{(k-1)} - n_2^{(k-1)}| = g^2 N \pi^2 \sum_{k=1}^{r} |n_1^{(k-1)}|, \tag{23}$$

where $n_i^{(k)}$ is the excitation number associated with the ith momentum fraction of the kth cyclic permutation, e.g. $\mathcal{C}^2|n_1, n_2, n_3\rangle$ yields $|n_1^{(2)} - n_2^{(2)}| = |n_3 - n_2| + |-n_2|$. This could be a useful method for similar integral equations, like the one associated with adjoint Dirac fermions recently tackled in [14].

First, let us clean up the spectrum by using the orientation symmetry $\mathcal{T}$ of the Hamiltonian, Eq. (15). Note that both symmetry operators act a bit awkwardly on

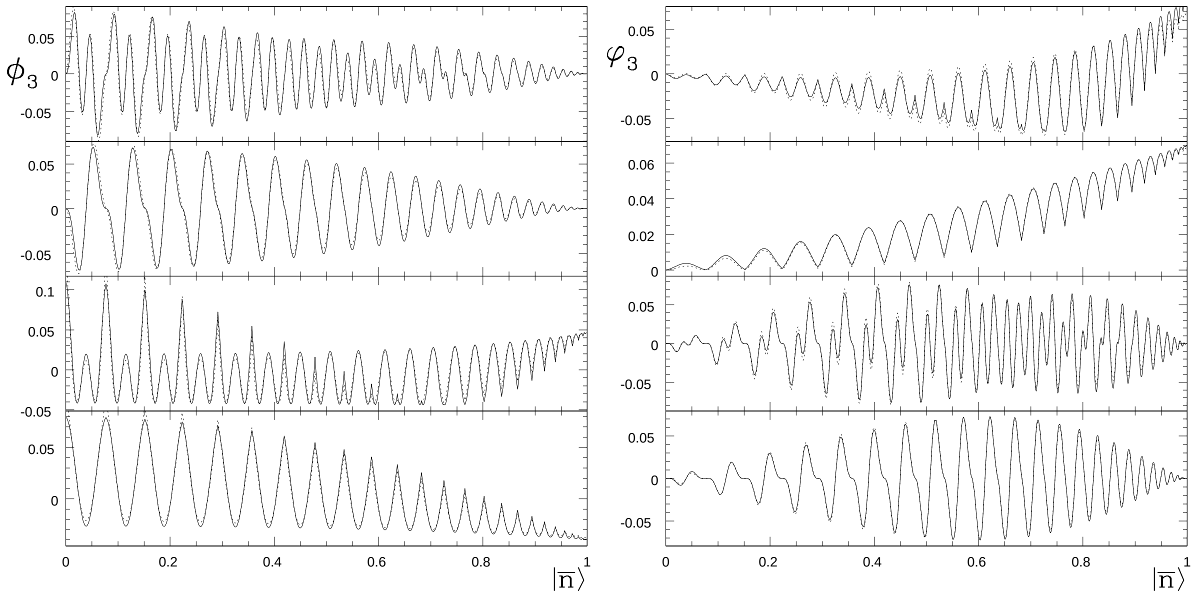

Figure 1: DLCQ eigenfunctions (solid lines) and asymptotic wavefunctions (dashed lines) of the theory without pair-production and non-singular terms. (a) Left: The lowest two three-parton eigenfunctions in the $\mathcal{T}$- even and -odd sectors (from bottom); $K = 151$ in the DLCQ calculation with massless fermions. (b) Right: Same for massive theory.

the basis states, as they are naturally defined with r variables, but actually live in a $(r-1)$-dimensional space

$$\begin{aligned} \mathcal{C} &: |n_1, n_2, \ldots, n_{r-1}\rangle \to (-1)^{n_{r-1}}| - n_{r-1}, n_1 - n_{r-1}, n_2 - n_{r-1}, \ldots, n_{r-2} - n_{r-1}\rangle, \\ \mathcal{T} &: |n_1, n_2, \ldots, n_{r-1}\rangle \to (-1)^{n_1}| - n_1, n_{r-1} - n_1, n_{r-2} - n_1, \ldots, n_2 - n_1\rangle, \end{aligned} \tag{24}$$

While $[\mathcal{C}^k, \mathcal{T}] \neq 0$, except for trivial cases, we have

$$\left[\sum_{k=1}^{r} \mathcal{C}^{k-1}, \mathcal{T}\right] = 0,$$

and, by construction, $[(-1)^{k(r-1)}\mathcal{C}^k, P^-] = 0$, so we can classify the eigenstates according to their eigenvalues M^2 and T. To fulfill the integral (eigenvalue) equation, one has to choose one specific $\mathcal{C}$ eigenvalue.

As a cross-check of our ansatz, we will compare to numerical wavefunctions generated by a DLCQ algorithm. A further check is provided by the solutions listed in Ref. [1], which can be emulated within DLCQ by choosing a large fermion mass, which enforces the constraint, Eq. (18) or (19), *vulgo* the vanishing of the wavefunction at the boundaries. We will refer to the latter solutions as massive parton solutions. We need to construct a complete orthonormal basis of the physical Hilbert space from the ansatz, Eq. (22). We'll work out the solutions in the first few sectors, and develop a general algorithm for the others.

At $r=2$ we have $\mathcal{C}|n\rangle = \mathcal{T}|n\rangle = (-1)^n|-n\rangle$, hence

$$\phi_2 = e^{i\pi n x} - (-1)^n e^{-i\pi n x}. \qquad (25)$$

Thus both sines with even n and cosines with odd n fulfill the integral equation, the cyclicity condition, and are states of definite T. Physics determines which functions to pick: massive partons require $\phi_2(0)=0$ or $\phi_2(x) = -\phi_2(1-x)$, whereas a massless theory requires odd n cosines, *i.e.* $\phi_2(x) = \phi_2(1-x)$, since a massless bound-state with a constant wavefunction exists in the limit $N_f \to 1$. We clean up the notation for the generic case, rewriting Eq. (25) as

$$|\phi_2, n; \bar{M}^2 = |n|\rangle_- = |n\rangle - (-1)^n|-n\rangle,$$

where $\bar{M}^2 = M^2/g^2N\pi^2$, and the minus index signifies that only the wavefunction odd under the $\mathcal{T}$ operation exists.

In the three-parton sectors, $r=3$, we find that both excitation numbers have to be even, because a massless bound state with constant wavefunction exists. For massive partons, no massless state exists, but the eigensolutions are again from the even-even $\{|ee\rangle\}$ sector, cf. Figure 1. The reason is that the $\mathcal{C},\mathcal{T}$ operators permute excitation numbers, *cf.* Eqs. (24), generating combinations like $n-m$ which are even for n,m odd.

The wavefunctions of definite $\mathcal{C},\mathcal{T}$ symmetry are

$$\begin{aligned} |\phi_3, n, m; \bar{M}^2 = |n-m| + |n| + |m|\rangle_\pm = \\ |n,m\rangle + (-1)^m|-m, n-m\rangle + (-1)^n|m-n,-n\rangle \\ \pm\left[(-1)^n|-n, m-n\rangle + |m,n\rangle + (-1)^m|n-m,-m\rangle\right], \end{aligned} \qquad (26)$$

which are symmetric $(+)$ or antisymmetric $(-)$ under reversal of momentum fractions. Note that some solutions do not exist in the $\mathcal{T}$-odd sector, e.g. $|\phi_3, n=-2, m=0; \bar{M}^2 = 4\rangle_- = 0$. The massless solution has a constant wavefunction with $n=m=0$.

Note that the states, Eq. (26), are not eigenfunctions of the Hamiltonian, because they do not fulfill Eq. (19). In order to create (anti-)symmetric wavefunctions we must combine positive and negative frequency solutions. This is natural, since the fermions are real, and hence the eigenfunctions can be chosen to be real. For instance

$$\begin{aligned} Re|\phi_3, n, m\rangle &= \cos(\pi n x_1 + \pi m x_2) + (-1)^m\cos(-\pi m x_1 + \pi(n-m)x_2) \\ &\quad +(-1)^n\cos(\pi(m-n)x_1 - \pi n x_2) \pm (\bar{n}_1 \leftrightarrow \bar{n}_2), \end{aligned}$$

where $\bar{n}_i$ is the excitation number associated with x_i in a term, e.g. $\bar{n}_1 = -m$ in the $(-1)^m$ term, which have to be permuted to obtain a state of definite symmetry under reversal of momentum fractions due to the $\mathcal{T}$ symmetry; the $(-1)^{n_i}$ factors remain unchanged. Note the disappearance of the $\pm$ sign: the real wavefunctions are all symmetric under momentum fraction reversal; the antisymmetric functions are identically zero. We can transcribe the wavefunction into (x_1,x_2,x_3) notation to obtain an expression manifestly symmetrized in the momentum fractions

$$\phi_{3+}^{(n,m)}(x_1,x_2,x_3) = \sum_{i=1}^{3}\cos(\pi n x_i + \pi m x_{i+1}) + (n \leftrightarrow m), \qquad (27)$$

The functions with the lowest excitation numbers are a decent fit to the lowest (DLCQ) eigenfunctions, *i.e.* $|1\rangle = \phi_{3+}^{(0,0)} = const.$, $|2\rangle = \phi_{3+}^{(2,0)} = \phi_{3+}^{(2,2)}$, $|3\rangle = \phi_{3+}^{(2,-2)} = \phi_{3+}^{(4,2)}$, $|4\rangle = \phi_{3+}^{(4,0)}$, $|5\rangle = \phi_{3+}^{(6,2)}$, see Fig. 1. From Eq. (26) it is clear that $\phi_{3\pm}^{(n,m)} = \pm\phi_{3\pm}^{(m,n)}$, and $\phi_{3+}^{(n,m)} = \phi_{3+}^{(-n,-m)}$, but note that in general distinct sets of excitation numbers do not result in distinct wavefunctions.

The odd T solutions are the imaginary part of the general wavefunction

$$\phi_{3-}^{(n,m)}(x_1, x_2, x_3) = \sum_{i=1}^{3} \sin(\pi n x_i + \pi m x_{i+1}) - (n \leftrightarrow m). \tag{28}$$

Again, the functions with the lowest excitation numbers are a decent fit to the lowest (DLCQ) eigenfunctions, *i.e.* $|1\rangle = \phi_{3-}^{(4,2)}$, $|2\rangle = \phi_{3-}^{(6,2)}$, $|3\rangle = \phi_{3-}^{(8,2)}$, $|4\rangle = \phi_{3-}^{(8,4)}$, see Fig. 1. Note that $\phi_{3-}^{(n,n)} = \phi_{3-}^{(n,0)} = 0$ and $\phi_{3-}^{(n,m)} = -\phi_{3-}^{(-n,-m)}$.

The massive parton solutions φ_3 are well-described by the same formulae in the opposite $\mathcal{T}$-sector, *i.e.*

$$\varphi_{3\pm}^{(n,m)}(x_1, x_2, x_3) = \phi_{3\mp}^{(n,m)}(x_1, x_2, x_3),$$

with the excitation numbers of the lowest eigenfunctions being $(2,0), (4,0), (6,2), (6,0)$ and $(6,2), (8,2), (10,4), (10,2)$, in the $\mathcal{T}$-even and $\mathcal{T}$-odd sectors, respectively. In the latter sector many functions are identically zero due to $\varphi_{3-}^{(n,m)} = -\varphi_{3-}^{(n,n-m)}$.

As a more stringent test of our ansatz we expanded the numerical solutions into the complete set of functions just derived, and checked that the coefficients of the expansion fall off fast. Note, though, that we are comparing numerical eigensolutions of the true, amputated[5] Hamiltonian, with analytic eigensolutions of the asymptotic Hamiltonian. Surprisingly, the eigenfunctions are perfectly reproduced with only a few non-vanishing coefficients, while the eigenvalues are off. For instance, at $r = 3$ ten basis states produce overlaps of larger than 99.5% with the first few eigenfunctions in the sector with the massless state, and the overlaps with the tenth function are in the permille range. The conclusion is that for low excitation numbers the mass renormalization term and the true integral limits are important to obtain the correct eigenvalues, whereas the symmetries of the system (cyclicity of the integral equation, orthogonality constraints of the physical Hilbert space) are so stringent that, assuming sinusoidal functions, there is very little leeway to choose the eigenfunctions, so they are basically fixed.

At $r = 4$, the states of definite $\mathcal{C}, \mathcal{T}$ symmetry are

$$\begin{aligned} &|\phi_4, n, m, l; \bar{M}^2 = |n-m| + |n| + |l| + |m-l|\rangle_\pm = \qquad (29)\\ &\quad |n, m, l\rangle - (-1)^l|-l, n-l, m-l\rangle + (-1)^m|l-m, -m, n-m\rangle - (-1)^n|m-n, l-n, -n\rangle \\ &\pm\Big[(-1)^n|-n, l-n, m-n\rangle - |l, m, n\rangle + (-1)^l|m-l, n-l, -l\rangle - (-1)^m|n-m, -m, l-m\rangle\Big] \end{aligned}$$

Due to the intricate way the excitation numbers are linked to the mass of the bound state, Eq. (23), states with distinct sets of excitation numbers may have identical masses. The (orthogonal) eigenstates of the Hamiltonian are thus linear combinations

[5] Correct integral boundaries are used, but parton-number violating terms have been chopped off.

of these states. For instance, the lightest states, with $\bar{M}^2 = 2(|n_1| + |n_2|)$ stem from the combination

$$\phi_{4-}(x_1, x_2, x_3, x_4) \doteq |\phi_4, n_1, 0, n_2\rangle - |\phi_4, n_1, 0, -n_2\rangle + |\phi_4, n_1, n_1 - n_2, -n_2\rangle, \quad (30)$$

which is the subset of solutions displayed as wavefunctions $\phi_4(x_1, x_2, x_3, x_4)$ in Ref. [1], Eq. (4.13). We find empirically that the excitation numbers are all even and that there are other solutions not describable by Eq. (30).

In summary, we see that the symmetrization of states with $r-1$ excitation numbers yields a surprisingly simple solution for adjoint QCD$_2$, and is in agreement with previous results, which turn out to be special cases of the general eigensolutions presented here.

4 The impact of non-singular operators

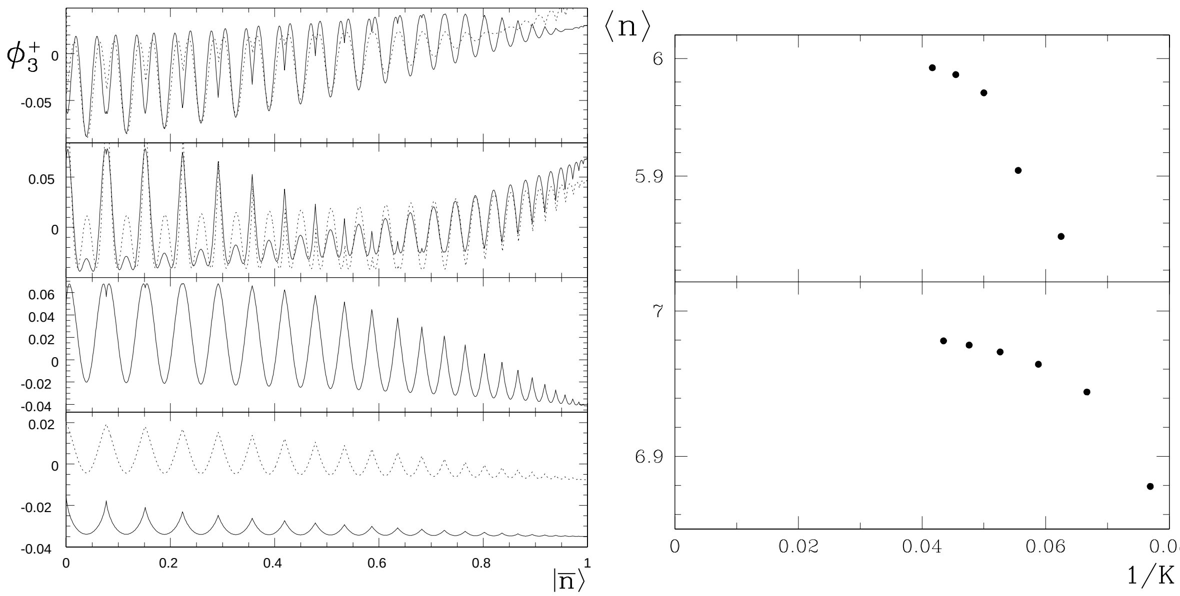

Figure 2: (a) Left: The lowest four three-parton $\mathcal{T}$-even DLCQ eigenfunctions of the theory with non-singular terms at $K = 151$ (solid lines) and of the asymptotic theory (dashed lines). Of the latter, the lowest eigenfunction has been suppressed by a factor five, and the second(third) lowest appears as analogue of the third(fourth) lowest non-singular function. (b) Right: Average parton number as a function of $1/K$ of a $\mathcal{T}$-even boson (top, $\epsilon = 0.225$, $M^2 \approx 41$) and fermion (bottom, $\epsilon = 0.505$, $M^2 \approx 31$).).

In Sec. 3 we solved for the spectrum of $P^-_{asympt} \equiv P^-_{PC,s} + P^-_{ren}$ keeping only singular terms in the Hamiltonian. While it will be hard to find analytic solutions without omitting non-singular operators, a numerical solution can be obtained without any problems. We find some noteworthy changes when regular operators are included.

The two-parton solutions are entirely unaffected by the regular terms, the lowest mass being $\bar{M}^2 = 11.74$. In contrast, the lowest $\mathcal{T}$-even three-parton mass jumps

dramatically, as the massless state acquires a mass (squared) of 5.703 when regular terms are present. Its wavefunction has the same structure as the lowest massive asymptotic one, save for an overall shift due to the admixture of the constant massless wavefunction. We note three things. Inclusion of non-singular terms inverts the mass hierarchy of massive states, namely a three-parton state becomes lighter than a two-parton state[6]. Secondly, this mass is very close to the continuum value obtain for the full theory $\bar{M}^2_{\mathrm{full,f}} = 5.75$, *cf.* $\bar{M}^2_{\mathrm{full,b}} = 10.84$ of the lightest boson. We infer that the lowest state is very pure in parton number, consistent with previous results [3]. Thirdly, the only two sectors unchanged by the inclusion of non-singular terms are the two-parton and the $\mathcal{T}$-odd three parton sectors.

Why is the $\mathcal{T}$-even three-parton sector so heavily influenced by non-singular operators? We can get the idea by studying the wavefunctions. In the $\mathcal{T}$-odd sector, the wavefunction is an odd function of the momenta, see Fig. 1(a) and Eq. (28). That means that the contributions from $1/(k_1 + k_2)^2$ in $A(k_i)$, Eqn. (7), will cancel and the masses will stay the same. In the $\mathcal{T}$-even sector, corrections are large when the wavefunction is large at the boundaries, *i.e.* where (at least) one momentum vanishes, e.g. $(0, x_2, x_3 = 1 - x_2)$. We would therefore expect the first, second and fourth $\mathcal{T}$-even *massive* eigenvalues to change substantially, but not the third, see the dashed wavefunctions in Fig. 2(a). To confirm our intuition, we compute the first and second corrections to the masses by sandwiching the operator $\langle i|2P^+P^-_{PC,r}|j\rangle$. It is easiest to do this numerically, using the existing eigensolutions of the asymptotic (unperturbed) Hamiltonian. We obtain for the lowest five masses

$$\begin{aligned}
\bar{M}_0^2 &= 0 + 5.961 - 0.3536 = 5.607(5.703) \\
\bar{M}_1^2 &= 21.59 + 8.589 - 1.564 = 28.62(29.05) \\
\bar{M}_2^2 &= 46.66 + 9.776 - 2.136 = 54.30(54.20) \\
\bar{M}_3^2 &= 56.07 + 1.662 + 0.3040 = 58.04(59.54) \\
\bar{M}_4^2 &= 74.29 + 10.92 - 1.562 = 83.65(83.81),
\end{aligned} \tag{31}$$

in agreement with our expectations. The non-perturbative results are listed in parentheses. Unsurprisingly, we find

$$\langle \phi^-_{3,i}|2P^+P^-_{PC,r}|\phi^-_{3,j}\rangle = 0$$

for all i, j, hence the corrections to the $\mathcal{T}$-odd eigenstates vanish identically.

Although we find a massless state in all $T = (-1)^{r+1}$ r-parton sectors, the $\mathcal{T}$-odd three-parton sector is the only one that does not receive corrections. The corrections are substantial in the other sectors (4+: 40%, 4−: 80%, 5+: 57%, 5−: 38%, 6+: 167%, 6−: 131%; at typical resolutions K). It is remarkable that the non-singular terms generate most of the mass of the six-parton states. Of course, none of this is in contradiction with the assumption that the asymptotic Hamiltonian is a good approximation at high excitation numbers.

[6]This is natural in the bosonized theory where the three parton state corresponds to the lowest state $\mathrm{Tr}\{J\psi\}|0\rangle$, see Sec. 6.

5 The role of parton-number violating operators

If we include the parton-number violating operators, we obtain the full theory at large N_c. Again a numerical solution can be obtained easily, with the caveat of a much higher number of basis states due to the coupling of parton sectors. Approximate (numerical) solutions are well-documented in the literature, see e.g. [9].

The parton-number changing interactions are three-body operators, and therefore have the largest influence on three-parton states. The relevant function, Eq. (8), is small when the momenta are roughly the same, and large when $k_1 - k_3$ is large while k_2 is small. Since $k_3 = 1 - k_1 - k_2$, the biggest contributions arise when k_1 and k_2 are very different.

We can investigate the role of parton-number violating operators using perturbation theory by parametrizing the Hamiltonian

$$P^- = P^-_{asympt} + \epsilon P^-_{PV}.$$

Previously, we argued on physical grounds that P^-_{PV} is marginal at high excitation numbers without explicitly identifying a small parameter. Here, we use ϵ to continuously switch from asymptotic to full theory. Obviously, the only non-zero parton-blocks of P^-_{PV} lie on its upper and lower secondary diagonals. Consequently, there is no first-order correction to the eigenvalues. At second order an r-parton eigenfunctions receives admixtures of $r-2$ and $r+2$ states only. In particular, the two(three)-parton eigenfunctions exhibit only four(five)-parton contaminations.

In Sec. 3 we found analytic expressions for a complete set of eigenfunctions of the asymptotic Hamiltonian. Hence, the matrix elements $\langle \phi_{r,\pm} | P^-_{PV} | \phi_{r+2,\pm} \rangle$ can in principle be calculated analytically. However, it should suffice to evaluate the operators numerically and extrapolate to the continuum, with the advantage of using *ab ovo* correct (at a certain K) solutions[7].

Empirically, we find that parton-number violation is necessary to produce (exact) multi-particle states in the spectrum. The reason is that only the complete Hamiltonian can be cast into a current-current form in the bosonized theory, where the decoupling of the multi-particle states can be seen explicitly [5].

We will use the purity of states in parton number as a measure of the importance of pair production. The authors of Refs. [3, 4] infer that the lowest states of the theory are very close to being eigenstates of the parton-number operator. Looking at the mass versus ϵ plot, Fig. 3(a), it appears that this is a by-product of the fact that the lowest states are quite isolated in mass. Consequently, these states are mostly inert with respect to admixtures from other parton sectors, and pair production is not important for the lowest states. However, since there are no (exact) multi-particle states without pair production, it has to be crucial for the other states. This importance may, however, not be reflected in parton-number impurity.

In Fig. 3(a), there are two distinct behaviors when the masses of two states are similar, $M_i(\epsilon) \approx M_j(\epsilon)$: either the eigenvalues repel or they are not influencing each

[7] In general, only a linear combination of the analytic asymptotic wavefunctions will be an eigensolution.

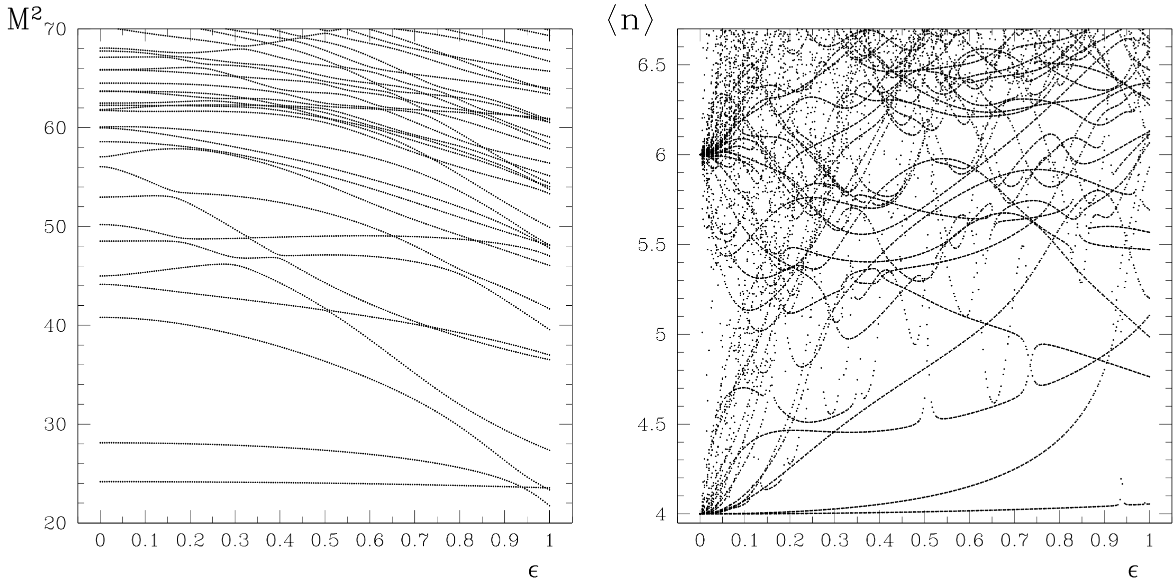

Figure 3: (a) Spectrum (b) average parton number of the states in the $\mathcal{T}$-odd bosonic sector as a function of the parton-number violation parameter ϵ.

other at all. To study these points, we plotted the average number of partons $\langle n\rangle$ in a state versus ϵ in Fig. 3(b). Although distinct trajectories of several states are discernible, the plots give us little leverage to decide which are the single-particle states. Whenever two states come close in mass, their other properties become similar, too. Although it is interesting to observe how some states "recover" from mixing at certain values of ϵ, the underlying message seems to be that states cannot be unambiguously identified as we continuously turn on pair production. It is a little disturbing that the function $\langle n\rangle(\epsilon)$ of the lowest $\mathcal{T}$-odd boson exhibits a cusp at $\epsilon \approx 0.94$. This seems to be a numerical artifact; the effect diminishes as K grows.

The lowest $\mathcal{T}$-odd fermion is a very pure five-parton state. This begs the question if the scheme[8] continues. We do see evidence for it. In particular, there is a pure six-parton state (up to $\epsilon \approx 0.3$ for $K = 24$), and there is a pure seven-parton state for $\epsilon < 0.6$. Both states are $\mathcal{T}$-even. We followed the development of these states at larger resolution, and they seem to stabilize. Namely, they become purer in parton number as ϵ and K grow, see Fig.2(b); the other states in this sector become less pure. If we extrapolate the curves $\langle n\rangle(\epsilon, K)$ towards the continuum, we obtain $\langle n\rangle = 6$ and $\langle n\rangle = 7$, respectively. This suggests the existence of a tower of infinitely many single-particle states organized in a double Regge trajectory.

[8]Lowest single-particle states in the bosonic $\mathcal{T}+$, the fermionic $\mathcal{T}+$, the bosonic $\mathcal{T}-$, and the fermionic $\mathcal{T}-$ sectors are pure 2,3,4,5-parton states, respectively.

6 Implications of Bosonization

The structure of the QCD$_{2A}$ spectrum is best understood in terms of current operators $J(-p) \sim \int dq\, b(q)b(p-q)$, *i.e.* by looking at the bosonized theory [11]. Fermionic states have an additional single fermion operator. The eigenvalues are the same as in the fermionic picture[9], yet the eigenfunctions are not, due to the fact that bosonization is a basis transformation. To find the single-particle states it is sufficient to restrict calculations to the single-trace sector [5]. As pointed out earlier, the problem is that not all single-trace states are single-particle states. Bosonization organizes the single-trace sector into blocks with distinct numbers of single-fermion operators ($f = 0, 1, 2, \ldots$), yet only the blocks with $f = 0, 1$ give rise to single-particle states [5]. The task to rid these blocks of (approximate) multi-particle states to reveal the true, single-particle content of the theory is hard. The problem is the mixing of the approximate multi-particle states with the single-particle states at any finite resolution.

We can quickly confirm that the above block-diagonalization is realized in any framework[10] with discrete momentum fractions. This exercise will make it easier to understand the role of the approximate multi-particle states by projecting out the exact multi-particle states. It requires the construction of direct-product (DP) states of the form

$$|DP\rangle = \mathrm{Tr}[J^{n_1}\psi J^{n_2}\psi \cdots J^{n_s}\psi]|0\rangle,$$

where J^{n_1} is a product of n_1 current operators carrying, in general, different (integer) momentum fractions, $\psi \equiv b(-1/2)$ is a fermion operator of momentum fraction $1/2$, and $s > 1$. Note that by constructing the DP states, we explicitly show that in QCD$_{2A}$ one cannot identify single-trace and single-particle states contrary to the 't Hooft model [2]. The important result of this exercise is that *only fermionic states* of the form $\mathrm{Tr}[J^n\psi]|0\rangle$ form exact multi-particle states, and therefore (likely) also the approximate MPS, as found numerically in [9, 10, 12].

The dimension of the DP sector of the bosonized single-trace sector plus the dimension of the (potential) single-particle sectors add up to the dimension of the single-trace sector in the fermion picture, for both the fermionic and the bosonic sectors of the theory. For instance, for $K = 21/2$ one has 1169 states in the fermionic picture, and 512 in the bosonized theory. Counting direct product states of the form $|K_1\rangle \otimes |K_2\rangle \otimes \cdots \otimes |K_s\rangle$ ($\sum_j K_j = 21/2$) one arrives at 697, but 40 states of the form $[|K = 7\rangle]^3$ are cyclically redundant, see Table 1. This implies that cyclic permutation of "constituent fermions" does not lead to independent states, consistent with the behavior of the bosonized states of the bosonic sector. For example, at $K = 4$, we have

$$\mathrm{Tr}[J(-2)\psi J(-1)\psi]|0\rangle = \mathrm{Tr}[J(-1)\psi J(-2)\psi]|0\rangle,$$

up to terms with a lesser number of operators. Pauli exclusion dictates that direct

[9] Although the expressions "bosonized theory" and "fermionic picture" appear on unequal footing, they help to avoid ambiguous expressions.

[10] We describe a DLCQ construction; analogous procedures exist whenever the spatial dimension is compactified.

product states of identical fermionic bound states, like $\mathrm{Tr}[\{J(-n)\psi\}^2]|0\rangle$, vanish[11]. We note that the number of DP states implies that *all* states, including the approximate multi-particle states, form DP states, while the much smaller number of approximate MPS (maximally the sum of the dimensions of the blocks with less than two single-fermion operators) suggests that only some, mostly likely the single-particle states, form those.

	Ferm.Pic.		Bos.Theory		DP		Ferm.Pic.		Bos.Theory		DP
$2K$	$T+$	$T-$	$T+$	$T-$	states	K	$T+$	$T-$	$T+$	$T-$	states
3	1	0	1	0	0	2	1	0	1	1	0
5	1	1	1	1	0	3	1	1	1	2	0
7	3	1	3	1	0	4	4	2	3	1	2
9	4	5	4	4	1	5	5	6	3	3	5
11	11	7	10	6	2	6	16	12	8	4	16
13	18	22	16	16	8	7	27	31	9	9	40
15	51	42	36	28	29	8	75	66	21	13	107
17	99	111	64	64	82	9	153	165	29	29	260
19	257	235	136	120	236	10	392	370	61	45	656
21	568	601	256	256	657	11	879	1791	93	93	1605
23	1421	1365	528	496	1762	12	2196	2142	191	159	3988
25	3312	3400	1048	1048	4664	13	5166	5254	315	315	9790
27	8209	8064	2080	2016	12177	14	12777	12632	622	558	24229

Table 1: Dimension of Fock bases in the fermionic picture and the bosonized theory. Fermionic (bosonic) states are on the left (right).

In sum, bosonization casts approximate and exact multi-particle states into different sectors. The hope is that this insight leads to a method to identify and eliminate the former from the $f = 0, 1$ blocks to reveal the true content of the theory.

7 Conclusions

We have constructed an algebraic solution of the asymptotic approximation to QCD_{2A} in the lowest parton sectors. We were able to elucidate the impact of non-singular parts of the Hamiltonian on the spectrum, and presented a perturbative calculation by smoothly turning on the parton-number violating operators. This allowed us to present evidence for the existence of two linear Regge trajectories of single-particle states, in accordance with earlier and recent work [12, 10].

While we were not able to find a criterion to distinguish single- from multi-particle states in general, we have presented several new facts that can be used towards finding the single-particle spectrum of QCD_{2A}. About the structure of the spectrum we learned the following: all states form exact multi-particle states, but only fermionic states with

[11] For the general rule, see [3], Sec. III.

exactly one fermionic operator form approximate multi-particle states. Furthermore, coupling between parton sectors is a necessary condition for the existence of multi-particle states.

Ref. [15] cautions us not to read too much into differences of approximations to the theory at finite resolution. On the other hand, the appearance of multi-particle states has been seen in two very different approaches [9, 10], and therefore hints at a framework-independent problem. In classic DLCQ the Hamiltonian is block-diagonal in resolution, but the supersymmetry operators are not, as the additional fermion has non-zero momentum at finite resolution. Hence, spurious interactions between single- and multi-particle states are induced to guarantee supersymmetry at $m_{SUSY} = g^2 N$ in the continuum limit, which make it hard to separate them. Even a manifestly supersymmetric framework like SDLCQ [16] does not circumvent the problem. The need to use periodic boundary conditions induces other interactions, and leads to worse convergence for massless fermions.

It may make sense to attempt to understand the spectrum of the theory using supersymmetry, which is exact for m_{SUSY} and 'softly' broken otherwise [17]. One idea is to flesh out the construction of wavefunctions by applying the supersymmetry generator sketched in [1]. This should work off the supersymmetric point [15] for the asymptotic theory.

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A Physical Hilbert Space

To solve the eigenvalue problem, Eq. (13), we need to use a basis of the physical Hilbert space. Due to the cyclic symmetry of states made of adjoint partons and the fixed, total momentum set to unity, we have an integration volume in the r-parton sector[12]

$$\int_0^{1/r} dx_1 \left(\prod_{i=2}^{r-1} \int_{x_1}^{1-(r-1)x_1-\sum_{j=2}^{i-1} x_j} dx_i \right) = \frac{1}{r!}.$$

It seems that we have singled out x_1 and x_r, but the wavefunctions are cyclic in all momentum fractions which eliminates this concern.

Our naive choice of states, the ansatz (20), is not orthogonal on the physical Hilbert space for $r > 3$, and thus constitutes an overcomplete basis. However, we can find linear combinations which group the naive solutions into $1/r!$ conjugacy classes orthogonal on the physical Hilbert space. For $r < 4$ we are done, because the $\mathcal{C}, \mathcal{T}$ operators exhaust the possibilities. For $r > 3$ we have to form linear combinations of $\frac{1}{2}(r-1)!$ states.

[12] We can ignore states eliminated by Pauli exclusion, since they constitute a set of measure zero.

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